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ON RIGHT CYCLICALLY ORDERED GROUPS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "ON RIGHT CYCLICALLY ORDERED GROUPS" submitted by BHALCHANDRA C. OLTIKAR in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

In this thesis we define right cyclically ordered groups and study some of their properties.

In Section I we show the connection between right cyclically ordered groups and right ordered groups.

In Section II we show that for periodic groups the two concepts cyclic order and right cyclic order coincide. We also show in this section that the infinite dihedral group can be right cyclically ordered. This serves as an example of a right cyclically ordered group that cannot be cyclically ordered. We also give a necessary and sufficient condition for the periodic elements of a right cyclically ordered group to form a subgroup.

In Section III we prove that the direct product of a right ordered group and a right cyclically ordered group is right cyclically ordered. From Swierczkowski's results [7] it follows that a torsion free cyclically ordered group is an ordered group. We give an example of a torsion free right cyclically ordered group that cannot be right ordered.

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INTRODUCTION

In [6] Rieger studied cyclically ordered groups which he defined as follows:

Definition: A group K is said to be cyclically ordered if for triplets x,y,z of distinct elements of K a ternary relation (x,y,z) is defined with the following properties:

C(1) Exactly one of (a,b,c) and (a,c,b) holds, where a,b,c are distinct elements of K.

C(2) $(a,b,c) \Rightarrow (b,c,a)$

C(3) (a,b,c) and $(a,c,d) \Rightarrow (a,b,d)$

C(4) $(a,b,c) \Rightarrow (xay,xby,xcy)$ for all $x,y \in K$.

Intuitively speaking, a cyclically ordered group is one whose points lie on a directed circle.

In our work we weaken C(4) to RC(4) :

$$(a,b,c) \Rightarrow (ax,bx,cx) \quad \text{for all} \quad x \in K .$$

More precisely we will study certain aspects of a right cyclically ordered groups which we define as:

Definition: A group K is called right cyclically ordered if for triplets x,y,z of distinct elements of K a ternary relation (x,y,z) is defined satisfying

$C(1)$, $C(2)$, $C(3)$ and $RC(4)$.

In the sequel we will need the following concepts:

A group is called an <u>0-group</u> if it admits full order. A subgroup C of an 0-group G is said to be <u>convex</u> if $x \in G$ and $c^{-1} < x < c$, $c \in C$ implies that $x \in C$. Let M be a subset of an 0-group G . We say that <u>G is convexly generated by M</u> to mean that the convex subgroup generated by M is G .

Let G be a group with identity e . Then G is called a <u>right ordered group</u> (ro-group) provided

(1) $(G,<)$ is an ordered set (linearly ordered)

(2) if $a < b$, then $ac < bc$ for all a,b,c in G .

A subgroup C of an ro-group G is said to be <u>convex</u> if $x \in G$ and $e < x < c$, $c \in C$ implies that $x \in C$. Let M be any subset of an ro-group G . We say that <u>G is convexly generated by M</u> to mean that the convex subgroup generated by M is G .

A group G is an R-group if for $a,b \in G$, $a^n = b^n$ (n positive integer) implies that $a = b$.

A group G is an R^*-group if $a^{x_1}\cdots a^{x_n} = b^{x_1}\cdots b^{x_n}$ implies that $a = b$, where $a,b,x_1,\cdots x_n \in G$ and $a^{x_i} = x_i^{-1}ax_i$, $i = 1,\cdots,n$.

Let G be any group. A subgroup H of G is called <u>isolated</u> in G if $a \in G$, $a^n \in H$ and n any positive integer implies that $a \in H$.

If H is any subgroup of G, then the <u>isolator</u> $I(H)$ of H is the intersection of all isolated subgroups in G containing H.

If M is any subset of G then $I(M) = I(\{M\})$, where $\{M\}$ denotes the subgroup generated by M.

In particular $z \in G$, then $I(z) = I(\{z\})$.

We observe that

$$\text{O-group} \subseteq R^{*}\text{-group} \subseteq \text{R-group}$$

and the centre of an R-group is isolated. (See Kurosh [4], p. 244).

In Section I we show that right cyclically ordered groups are related to ro-groups exactly as cyclically ordered groups are related to O-groups.

It is known (see Fuchs [3], p. 64) that a cyclically ordered group K is isomorphic to $G/\{z\}$ where G is an O-group, z is in the centre of G $(z > e)$, and G is convexly generated by z.

Analogously we have obtained the following result for right cyclically ordered groups:

<u>Theorem α</u>: A right cyclically ordered group is isomorphic to $G/\{z\}$ where G is an ro-group, z is in the centre of G $(z > e)$, and G is convexly generated by z.

In Section II, we show that the concept of the cyclic order and that of the right cyclic order coincide in the case of periodic groups.

Now it is known that the periodic elements of a cyclically ordered group form a subgroup of the centre (see Fuchs [3], p. 63). However by making use of the theorem α , we can show that the infinite dihedral group can be right cyclically ordered, which in turn shows that the periodic elements of a right cyclically ordered group need not form a subgroup. This naturally leads to a question as to when the periodic elements in a right cyclically ordered group will form a subgroup. We have the following answer to this question:

<u>Theorem β</u>: The periodic elements of a right cyclically ordered group K form a subgroup if and only if the isolator of z is contained in the centre of G , where G and z are as in the theorem α .

In Section III, we obtain some results on direct products of ro-groups and right cyclically ordered groups.

Swierczkowski [7] has proved that the direct product of an O-group and a cyclically ordered group is a cyclically ordered group. Further, every cyclically ordered group is isomorphic to a subgroup of the direct product of some ordered group Γ and the group C of complex roots of unity under multiplication.

We prove that the direct product of an ro-group and a right cyclically ordered group is a right cyclically ordered group. However, we give an example to show that a right cyclically ordered group may not be isomorphic to any subgroup of the direct product of an ro-group Γ and the group C of complex roots of unity under multiplication. (Even when the periodic part of right cyclically ordered group forms a subgroup.)

SECTION I

Definition 1.1: A group K is said to be right cyclically ordered if for triplets x,y,z of distinct elements of K a ternary relation (x,y,z) is defined with the following properties:

RC(1) Exactly one of (a,b,c) and (a,c,b) holds where a,b,c are distinct elements of K .

RC(2) $(a,b,c) \Rightarrow (b,c,a)$

RC(3) (a,b,c) and $(a,c,d) \Rightarrow (a,b,d)$

RC(4) $(a,b,c) \Rightarrow (ax,bx,cx)$ for every $x \in K$.

Remark: From (a,b,c) using RC(2) twice, we get (c,a,b) and from (a,c,d) we get (c,d,a) . Thus

$$\left.\begin{matrix}(a,b,c)\\(a,c,d)\end{matrix}\right\} \Rightarrow \left.\begin{matrix}(c,a,b)\\(c,d,a)\end{matrix}\right\} \Rightarrow \text{(by RC(3))} \quad (c,d,b)$$

$$\Rightarrow (b,c,d) \quad .$$

Definition 1.2: $(a_1,a_2,\cdots,a_n)$ $n \geq 3$ means that (a_i,a_j,a_k) for all $i < j < k$ $(1 \leq i,j,k \leq n)$.

Lemma 1.3: If (a,b_i,b_{i+1}) , $i = 1,2,\cdots,n-1$, then $(a,b_1,\cdots,b_n)$, $n \geq 2$.

Proof: We shall prove the lemma by induction. For $n = 2$ the result is

true by hypothesis. Assume the result for $n = k$, that is,

$$(a,b_i,b_{i+1}) \quad , \quad i = 1,2,\cdots,k-1 \Rightarrow (a,b_1,\cdots,b_k) \quad .$$

Assume further that (a,b_k,b_{k+1}) .

<u>Claim</u>: $(a,b_1,\cdots,b_{k+1})$.

For $i < j \leq k$, (a,b_i,b_j) holds by the induction hypothesis. So let $j = k+1$ and $1 \leq i < k+1$. If $i = k$, then by assumption (a,b_i,b_{k+1}) . If $i < k$, then (a,b_i,b_k) and $(a,b_k,b_{k+1}) \Rightarrow (a,b_i,b_{k+1})$. Thus (a,b_i,b_j) for all $i < j \leq k+1$. Also we must show that (b_i,b_j,b_h) for all $i < j < h$, $1 \leq i,j,h \leq k+1$. Again, if $h \leq k$, then (b_i,b_j,b_h) by induction hypothesis. Hence let $h = k+1$. Since we have (a,b_i,b_j) and (a,b_j,b_h) , by the remark after definition 1.1 (b_i,b_j,b_h) holds. The proof of the lemma is thus complete by induction.

<u>Remark</u>: In particular $a,b_1,\cdots,b_n$ are distinct.

<u>Theorem 1.4</u>: If a finite group G is right cyclically ordered, then G must be cyclic.

<u>Proof</u>: G surely contains an element $a \neq e$ such that no $x \in G$ satisfies (e,x,a) . For, otherwise we can construct a sequence $a = a_1 , a_2 , a_3 , \ldots$ such that (e,a_{i+1},a_i) which contradicts the lemma 1.3 in view of the finiteness of G .

If this a is of order 2 and (e,a,y) holds for some $y \in G$ then (a,a^2,ya) , that is, (a,e,ya) and hence (e,ya,a) holds giving us a contradiction. Thus G contains no nonidentity element other than a

and $G = \{a\}$. (Note that the group of order 2 satisfies all the axioms of the definition 1.1 vacuously since it does not contain three distinct elements.)

If a is of order $n \geq 3$, then (e,a,a^2) holds, which implies (a^k,a^{k+1},a^{k+2}) for every integer k .

It is impossible to have (a^k,x,a^{k+1}) for any $x \in G$ and integer k .

For, $(a^k,x,a^{k+1}) \Rightarrow (e,xa^{-k},a)$. Also, (a^k,a^{k+1},x) for every integer k and $x \in G$ contradicts the lemma 1.3.

Hence $G = \{a\}$ and $(e,a,\cdots,a^{n-1})$.

1.5 Examples:

(1) Every cyclically ordered group is right cyclically ordered.

(2) Let G be right ordered group. G may be right cyclically ordered by defining:

$$(a,b,c) \text{ if and only if } a < b < c \text{ or } b < c < a \text{ or } c < a < b \;;\; a,b,c \in G .$$

This right cyclic order is called the induced order from its right order.

(3) Let G be a right ordered group containing an element z in the centre such that $z > e$ and G is convexly generated by z (in the sense of the definition of Conrad [2] p. 270) viz.

Definition: Let G be a right ordered group. A subgroup C of G is

said to be <u>convex</u> if $x \in G$ and $e < x < c$ and $c \in C$ imply that $x \in C$. We say that G is convexly generated by z to mean that the convex subgroup generated by z is G. Note that G is convexly generated by z means that for every $g \in G$ there is an integer $n = n(g)$ such that $g < z^n$.)

Let $K = G/\{z\}$. Then K can be right cyclically ordered by defining:

$$\begin{aligned}(\bar{a},\bar{b},\bar{c}) \quad \text{if and only if} \quad & r_a < r_b < r_c \ , \\ \text{or} \quad & r_b < r_c < r_a \ , \\ \text{or} \quad & r_c < r_a < r_b \ ,\end{aligned}$$

where $\bar{a},\bar{b},\bar{c}$ are distinct elements of K and r_a, r_b, r_c are the unique coset representatives of $\bar{a},\bar{b},\bar{c}$ respectively, satisfying

$$e \leq r_a, r_b, r_c < z \ .$$

In order to see that this defines a right cyclic order, first we show that for every coset $\bar{a}$ there is a unique coset representative r_a such that $e \leq r_a < z$. Let a be any representative of $\bar{a}$.

<u>Claim</u>: There is a k such that $z^{k-1} \leq a < z^k$.

<u>Proof of the claim</u>: Clearly, there is an $n = n(a)$ such that $a < z^n$. If $a < z^n$ for every integer n, then $e < z^n a^{-1} = a^{-1} z^n$ and hence $z^{-n} < a^{-1}$ for every integer n, which contradicts that G is convexly generated by z. Hence the claim.

Now set $r_a = az^{1-k}$. Then r_a is unique.
For, if not, let x,y be two distinct elements of G such that

$e \leq x$, $y < z$ and $\bar{x} = \bar{y}$. Without loss of generality we may assume that $x < y$. Thus $e < yx^{-1}$. Now $\bar{x} = \bar{y} \Rightarrow yx^{-1} \in \{z\} \Rightarrow yx^{-1} = z^k$, and $yx^{-1} > e \Rightarrow k > 0$. Hence $z^k = e \cdot z^k < x \cdot z^k = z^k x = y < z$, giving $k = 0$, that is, $x = y$ – a contradiction.

Now define

$$(\bar{a},\bar{b},\bar{c}) \Leftrightarrow r_a < r_b < r_c \quad ,$$
$$\text{or} \quad r_b < r_c < r_a \quad ,$$
$$\text{or} \quad r_c < r_a < r_b \quad .$$

It is immediate from the definition that the conditions RC(1) and RC(2) hold. Now let $(\bar{a},\bar{b},\bar{c})$ and $(\bar{a},\bar{c},\bar{d})$. Assume $(\bar{a},\bar{b},\bar{c})$ with $r_a < r_b < r_c$. Now

$$(\bar{a},\bar{c},\bar{d}) \Leftrightarrow \text{(i)} \quad r_a < r_c < r_d \quad ,$$
$$\text{or} \quad \text{(ii)} \quad r_c < r_d < r_a \quad ,$$
$$\text{or} \quad \text{(iii)} \quad r_d < r_a < r_c \quad .$$

Out of (i), (ii) and (iii), (ii) is not possible as $r_a < r_c$. So we must have either

(1) $r_a < r_b < r_c < r_d$

or

(2) $r_d < r_a < r_b < r_c$.

In either case $(\bar{a},\bar{b},\bar{d})$ holds. The other cases may be verified similarly. This establishes RC(3) .

Now, again $(\bar{a},\bar{b},\bar{c})$ holds and $\bar{x} \in K$.

Let r_x be the unique coset representative of $\bar{x}$. Now

$$r_{ax} = r_a r_x \quad \text{if} \quad r_a r_x < z$$

$$= r_a r_x z^{-1} \quad \text{if} \quad r_a r_x \geq z .$$

Assume $(\bar{a},\bar{b},\bar{c})$ holds with $r_a < r_b < r_c$,

<u>case (i)</u> when $r_{ax} = r_a r_x z^{-1}$. Since $r_a < r_b < r_c$ and $z \leq r_a r_x$, we have $z \leq r_a r_x < r_b r_x < r_c r_x$ which implies

$$r_{ax} < r_{bx} < r_{cx} \quad \text{and} \quad (\overline{ax},\overline{bx},\overline{cx})$$

holds.

<u>Case (ii)</u> when $r_{ax} = r_a r_x$ but $r_{bx} = r_b r_x z^{-1}$. Since $r_b < r_c$ and $z \leq r_b r_x < r_c r_x$, we get $r_{cx} = r_c r_x z^{-1}$. Thus $r_{bx} < r_{cx}$. Further, $r_{cx} < r_{ax}$. For, if not,

$$r_{ax} = r_a r_x < r_{cx} = r_c r_x z^{-1} ,$$

then

$$r_a r_x z = r_a z r_x < r_c r_x$$

which implies $z \leq r_a z < r_c$ - a contradiction. Thus we have $r_{bx} < r_{cx} < r_{ax}$ which implies $(\overline{ax},\overline{bx},\overline{cx})$.

<u>Case (iii)</u> when $r_{ax} = r_a r_x$, $r_{bx} = r_b r_x$ but $r_{cx} = r_c r_x z^{-1}$. In this case $r_{ax} < r_{bx}$. By case (ii) $r_{cx} < r_{ax}$. Thus $r_{cx} < r_{ax} < r_{bx}$ and

hence $(\overline{ax},\overline{bx},\overline{cx})$.

<u>Case (iv)</u> when $r_{ax} = r_a r_x$, $r_{bx} = r_b r_x$, $r_{cx} = r_c r_x$. Here $r_{ax} < r_{bx} < r_{cx}$ and again $(\overline{ax},\overline{bx},\overline{cx})$ holds.

The other cases may be verified similarly. Thus RC(4) is verified.

Hence $G/\{z\}$ is right cyclically ordered.

The next result shows that the above construction is the most general one.

<u>Theorem 1.6</u>: To every right cyclically ordered group K there is a right ordered group G and an element z in the centre of G such that K may be obtained from G as described in the example 3 of 1.5.

<u>Proof</u>: We define G as the central Schreier extension of an infinite cyclic group $\{z\}$ by the given group K .

More precisely, let G consist of all pairs $\langle z^n,a\rangle$ with integers n and $a \in K$, subject to the rules:

$$\langle z^n,a\rangle = \langle z^m,b\rangle \iff n = m \quad \text{and} \quad a = b$$

and

$$\langle z^n,a\rangle\langle z^m,b\rangle = \langle z^{n+m}f_{a,b},ab\rangle$$

where factors $f_{a,b}$ are defined by

$$f_{a,b} = \begin{cases} e & \text{if } a = e \text{ or } b = e \text{ or } (e,b,ab) \\ z & \text{if } ab = e \ (a \neq e) \text{ or } (e,ab,b) \ . \end{cases}$$

First we verify the associativity of the operation:

$$\begin{aligned} \langle z^{\ell},a\rangle[\langle z^{m},b\rangle\langle z^{n},c\rangle] &= \langle z^{\ell},a\rangle\langle z^{m+n}f_{b,c},bc\rangle \\ &= \langle z^{\ell+m+n}f_{a,bc}f_{b,c},a(bc)\rangle \end{aligned}$$

$$\begin{aligned} [\langle z^{\ell},a\rangle\langle z^{m},b\rangle]\langle z^{n},c\rangle &= \langle z^{\ell+m}f_{a,b},ab\rangle\langle z^{n},c\rangle \\ &= \langle z^{\ell+m+n}f_{a,b}f_{ab,c},(ab)c\rangle \quad . \end{aligned}$$

Thus associativity condition reduces to

$$f_{b,c}f_{a,bc} = f_{a,b}f_{ab,c} \qquad *$$

<u>Verification of * </u>:

<u>case (i)</u>: When at least one of a,b and c is identity.

If $a = e$, then both sides of * are equal to $f_{b,c}$.

If $b = e$, then both sides of * are equal to $f_{a,c}$.

If $c = e$, then both sides of * are equal to $f_{a,b}$.

<u>case (ii)</u>: When none of a,b,c is identity but $ab = bc = e$, Here both sides of * are equal to z .

<u>Case (iii)</u>: When a,b,c as in (ii) but only one of ab and bc is identity.

If $ab = e$ then $f_{a,b}f_{ab,c} = z$. Now either (e,c,bc) or (e,bc,c) . If (e,c,bc) , then $f_{b,c} = e$ and since $ab = e$, (e,c,bc) is nothing but (e,abc,bc) giving $f_{a,bc} = z$. Hence $f_{b,c}f_{a,bc} = z$. If (e,bc,c) then $f_{b,c} = z$ and (e,bc,c) is (e,bc,abc) (since $ab = e$) giving $f_{a,bc} = e$. Hence $f_{b,c}f_{a,bc} = z$.

Thus when $ab = e$, * holds.

Now if $bc = e$, then $f_{b,c}f_{a,bc} = z$. Again, either (e,b,ab) or (e,ab,b) . If (e,b,ab) , then $f_{a,b} = e$, and

$$(e,b,ab) \Rightarrow (c,bc,abc) ,$$

that is, (c,e,a) . But

$$(c,e,a) \Rightarrow (e,a,c)$$

which is nothing but (e,abc,c) (since $bc = e$) . But $(e,abc,c) \Rightarrow f_{ab,c} = z$. Hence $f_{a,b}f_{ab,c} = z$. If (e,ab,b) , then $f_{a,b} = z$. Also $(e,ab,b) \Rightarrow (c,abc,bc)$, that is, (c,a,e) , giving (e,c,a) . Since $bc = e$, $(e,c,a) \Rightarrow (e,c,abc) \Rightarrow f_{ab,c} = e$. Hence $f_{a,b}f_{ab,c} = z$. Thus when $bc = e$, * is verified.

<u>Case (iv)</u>: None of a,b,c,ab and bc is identity. There are four possibilities to be considered:

$$P(1): \quad (e,b,ab) \quad \text{and} \quad (e,c,bc)$$

$$P(2): \quad (e,ab,b) \quad \text{and} \quad (e,c,bc)$$

$$P(3): \quad (e,b,ab) \quad \text{and} \quad (e,bc,c)$$

$$P(4): \quad (e,ab,b) \quad \text{and} \quad (e,bc,c) \quad .$$

In P(1) we must show that

$$f_{a,bc} = f_{ab,c} \quad , \quad \text{since} \quad f_{a,b} = f_{b,c} = e \quad .$$

This is so if and only if

$$'\ (e,bc,abc) \Leftrightarrow (e,c,abc)\ ' \quad .$$

Let (e,bc,abc) . Also we have (e,c,bc) . Now (e,c,bc) and $(e,bc,abc) \Rightarrow (e,c,abc)$. Next assume (e,c,abc) . From (e,b,ab) we get (c,bc,abc) , that is, (abc,c,bc) . Also $(e,c,abc) \Rightarrow (abc,e,c)$. Hence (e,c,abc) and $(e,b,ab) \Rightarrow (abc,e,bc) \Rightarrow (e,bc,abc)$. Hence in P(1) , * holds.

In P(2) , we have (e,ab,b) and (e,c,bc) . Thus $f_{a,b} = z$ and $f_{b,c} = e$. Hence $f_{b,c}f_{a,bc} = f_{a,bc}$ and $f_{a,b}f_{ab,c} = zf_{ab,c}$. Now $(e,ab,b) \Rightarrow (c,abc,bc)$ and $(e,c,bc) \Rightarrow (c,bc,e)$. But (c,abc,bc) , $(c,bc,e) \Rightarrow (c,abc,e)$. Hence (e,ab,b) and $(e,c,bc) \Rightarrow (e,c,abc) \Rightarrow f_{ab,c} = e$. Hence $f_{a,b}f_{ab,c} = z$. Also from (bc,c,abc) and (bc,e,c) we get (bc,e,abc) .

$$(bc,e,abc) \Rightarrow (e,abc,bc) \Rightarrow f_{a,bc} = z \ .$$

Hence $f_{b,c}f_{a,bc} = z = f_{a,b}f_{a,bc}$.

Hence in P(2) * holds.

In P(3) , we have (e,b,ab) and (e,bc,c) . Thus $f_{a,b} = e$ and $f_{b,c} = z$.

Hence $f_{a,b}f_{ab,c} = f_{ab,c}$ and $f_{b,c}f_{a,bc} = zf_{a,bc}$. Now

$$(e,b,ab) \Rightarrow (c,bc,abc)$$

and

$$(e,bc,c) \Rightarrow (c,e,bc) \quad .$$

But (c,e,bc) and (c,bc,abc) => (c,e,abc) => (e,abc,c) => $f_{ab,c} = z$.

Hence $f_{a,b}f_{ab,c} = z$.

Again, from (bc,c,e) and (bc,abc,c) we get (bc,abc,e) , that is, (e,bc,abc) which implies $f_{a,bc} = e$. Hence $f_{b,c}f_{a,bc} = z$.

Hence $f_{a,b}f_{ab,c} = z = f_{b,c}f_{abc}$.

Hence in P(3) * holds.

In P(4) , we have (e,ab,b) and (e,bc,c) . Thus $f_{a,b} = f_{b,c} = z$. Hence we must show that $f_{ab,c} = f_{a,bc}$. But this is so if and only if

$$' \ (e,c,abc) \Leftrightarrow (e,bc,abc) \ ' \quad .$$

Assume (e,c,abc) . Also we have (e,bc,c)·(e,bc,c) and (e,c,abc) => (e,bc,abc) .

Next assume (e,bc,abc) . From (e,ab,b) we get (c,abc,bc) . But (abc,e,bc) and $(abc,bc,c) \Rightarrow (abc,e,c) \Rightarrow (e,c,abc)$. Thus $(e,c,abc) \Leftrightarrow (e,bc,abc)$. Hence in $P(4)$ $*$ holds.

Thus $*$ holds in all possible cases. Hence the operation in G is associative. Hence G is a group.

G can be right ordered by defining positive elements to be:

$$P = \{\langle z^n, a\rangle : n > 0 \text{ , or , } n = 0 \text{ and } a \neq e\} \text{ .}$$

This follows by (1), (2), (3) below:

(1) Clearly $\langle e,e\rangle \notin P$.

(2) Also $P\,P \subseteq P$.

For, if $\langle z^n,a\rangle$ and $\langle z^m,b\rangle \in P$, then

$$\langle z^n,a\rangle\langle z^m,b\rangle = \langle z^{n+m}f_{a,b},ab\rangle \in P \text{ .}$$

(3) If $g = \langle z^\alpha,a\rangle \in P$, then $\alpha \geq 0$. But then $g^{-1} = \langle z^{-(\alpha+1)},a^{-1}\rangle$ and hence g^{-1} cannot belong to P since $\alpha+1 > 0$.

Thus P is a positive cone for a right order of G .

However, in fact P turns out to be a full right order. For, if $g \neq e$, $g \in G$, $g \notin P$. Then $g = \langle z^\alpha,a\rangle$ and $\alpha < 0$. Hence $g^{-1} = \langle z^{-(\alpha+1)},a^{-1}\rangle$ $(a \neq e)$ and $-(\alpha+1) \geq 0$. Clearly $g^{-1} \in P$. If $a = e$, then $g^{-1} = \langle z^{-\alpha},e\rangle$. Since $-\alpha > 0$, $g^{-1} \in P$.

Thus G is a right ordered group. It is clear that $\langle z,e \rangle$ lies in the centre of G.

Also G is convexly generated by $\langle z,e \rangle$. For, if $g \in G$ and $g = \langle z^{\alpha},a \rangle$, then $g < \langle z,e \rangle^{\alpha+1}$. [This holds as follows:

If $a \neq e$, then
$$\langle z,e \rangle^{\alpha+1} g^{-1} = \langle z^{\alpha+1},e \rangle g^{-1}$$
$$= \langle z^{\alpha+1},e \rangle \langle z^{-(\alpha+1)},a^{-1} \rangle$$
$$= \langle e,a^{-1} \rangle \in P \ .$$

Hence $g < \langle z,e \rangle^{\alpha+1}$. If $a = e$, then

$$\langle z,e \rangle^{\alpha+1} g^{-1} = \langle z^{\alpha+1},e \rangle g^{-1}$$
$$= \langle z^{\alpha+1},e \rangle \langle z^{-\alpha},e \rangle$$
$$= \langle z,e \rangle \in P \ .$$

Hence $g < \langle z,e \rangle^{\alpha+1}$ in this case also.]

Next $G/\{\langle z,e \rangle\} \cong K$. For, firstly, every coset $\overline{a}$ mod $\{\langle z,e \rangle\}$ may be represented by a unique element $\langle e,a \rangle$ [This follows as:

First $\overline{\langle e,a \rangle} = \overline{\langle e,b \rangle} \Rightarrow \langle e,a \rangle^{-1} \langle e,b \rangle \in \{\langle z,e \rangle\}$,

that is,

$$\langle z^{-1},a^{-1} \rangle \langle e,b \rangle = \langle z^{-1} f_{a^{-1},b}, a^{-1}b \rangle \in \{\langle z,e \rangle\} \ .$$

This implies $a^{-1}b = e \Rightarrow a = b$. Thus $\overline{\langle e,a \rangle} = \overline{\langle e,b \rangle} \Rightarrow \langle e,a \rangle = \langle e,b \rangle$.
Next $\overline{\langle z^{\alpha},x \rangle} = \overline{\langle e,x \rangle}$, α any integer. For,

$$\langle z^{\alpha},x\rangle^{-1}\langle e,x\rangle = \langle z^{-(\alpha+1)},x^{-1}\rangle\langle e,x\rangle = \langle z^{-(\alpha+1)}\cdot z,e\rangle$$

$$= \langle z,e\rangle^{-\alpha} \in \{\langle z,e\rangle\} \quad .$$

Now we have to put for $e \neq a \neq b \neq e$

$$\begin{aligned}(\bar{e},\bar{a},\bar{b}) &\Leftrightarrow \langle e,a\rangle < \langle e,b\rangle \\ &\Leftrightarrow \langle e,b\rangle\langle e,a\rangle^{-1} \in P \\ &\Leftrightarrow \langle e,b\rangle\langle z^{-1},a^{-1}\rangle = \langle z^{-1}f_{b,a^{-1}},ba^{-1}\rangle \in P \\ &\Leftrightarrow (e,ba^{-1},a^{-1}) \\ &\Leftrightarrow (a,b,e) \\ &\Leftrightarrow (e,a,b) \quad .\end{aligned}$$

This completes the proof.

<u>Remark</u>: First consider an ro-group G with the semigroup P of positive elements. If we define $a \dashv b$ if and only if $a^{-1}b \in P$, then it is easy to see that:

(i) $(G,\dashv)$ is an ordered set.

(ii) If $a \dashv b$, then $ca \dashv cb$ for all $a,b,c \in G$.

Thus to P there corresponds a left ordering on G . Conversely if we start with a left ordered group G , the similar construction gives a right ordering on G .

Now given a right cyclically ordered group K , it is isomorphic to $G/\{z\}$, where G is a suitable ro-group. Using a corresponding left ordering on G , as described in the previous paragraph, one gets a left cyclic order on K . This eliminates the need to distinguish between the left and the right cyclic orders. Hence we are justified in discussing only the right cyclic order throughout.

SECTION II

We start here with some definitions and results from Conrad [2].

Definition 2.1: An ro-group G is archemedian if for every pair a,b in the positive cone P of G there exists a positive integer n such that $a < b^n$.

Lemma 2.2: (Conrad [2], Theorem 3.8, p. 271).

If an ro-group G is archemedian then G is an 0-group. Thus G is o-isomorphic to a subgroup of the additive group R of real numbers.

Theorem 2.3: Every periodic right cyclically ordered group is abelian.

Proof: Let K be periodic right cyclically ordered group.

From Theorem 1.6

$$K \cong G/\{z\} \quad ,$$

where G is an ro-group, $z \in Z(G)$, $z > e$ and G is convexly generated by z .

Claim: G is an archemedian ro-group. First consider elements of the type $\langle e,a\rangle$, $\langle e,b\rangle \in P$ - the positive cone of G so that $a \neq e$, $b \neq e$.

Let b be of order m .

If for some integer $n < m$ we get the first component of $<e,b>^n$ to be z , then clearly for this n we have

$$<e,a> < <e,b>^n \quad .$$

If for no integer $n < m$, the first component of $<e,b>^n$ is z , then

$$<e,b>^m = <z,e> > <e,a> \quad .$$

Now consider $<z^k,a>$, $<z^\ell,b> \in P$ such that $k > 0$ and $\ell > 0$. First observe that

$$<z^k,a> < <z^\alpha,x>$$

whenever $\alpha \geq k+1$, $x \in K$.

Since ℓ is a positive integer we can choose n such that $n\ell \geq k+1$ and for this n we have $<z^k,a> < <z^\ell,b>^n$. If $k = 0$ and $\ell > 0$, then clearly

$$<z^k,a> = <e,a> < <z^\ell,b> \quad .$$

If $k > 0$ and $\ell = 0$, then again we can find an integer n such that

$$<e,b>^n = <z^m,x> \quad ,$$

where m is a positive integer and then this reduces to the case which is

already discussed above. Hence the claim.

Thus by the lemma 2.2, G, being an archemedian ro-group, is 0-isomorphic to a subgroup of the additive group R of real numbers. In particular, G is abelian. Thus $K \cong G/\{z\}$ is abelian.

Definition 2.4: A right cyclically ordered group is said to be archemedian if it contains no elements x,y such that (e,x^n,y) holds for every positive integer n.

Theorem 2.5: If a right cyclically ordered group K is archemedian, then K is abelian.

Proof: For any $b \neq e$ in K, let $B = \{b\}$. Then K induces a right cyclic order on B that makes B archemedian. But B is abelian so that the right cyclic order on B is a cyclic order and hence B is periodic. (See [7] Swierczkowski, p. 162).

This proves that K is periodic and hence by Theorem 2.3 K is abelian.

The concepts of cyclic order and right cyclic order are same for periodic groups, but this is not so in general. This can be seen from

2.6 Example: The infinite dihedral group can be right cyclically ordered.

Let $G = \{a,b : b^{-1}ab = a^{-1}\}$. Then G is an ro-group with positive elements:

$$P = \{a^k b^\ell : \ell > 0 \text{ , or , } \ell = 0 \text{ and } k > 0\} .$$

Note that every $g \in G$ can be written as $a^n b^m$ where n and m are integers, since $ab = ba^{-1}$ and $ba = a^{-1}b$.

Now we show that P defines a right order on G . Firstly, P is subsemigroup of G . For, if $x,y \in P$, $x = a^{k_1}b^{\ell_1}$, $y = a^{k_2}b^{\ell_2}$, then $xy = a^{k_1}b^{\ell_1}a^{k_2}b^{\ell_2}$.

If ℓ_1 is odd, then

$$xy = a^{k_1-k_2}\,b^{\ell_1+\ell_2} = a^k b^\ell \qquad (\text{where } k = k_1 - k_2,\ \ell = \ell_1 + \ell_2)$$

since $\ell_1 > 0$, $\ell = \ell_1+\ell_2 > 0$ and $xy \in P$.

If ℓ_1 is even, then

$$xy = a^{k_1}b^{\ell_1}\,a^{k_2}b^{\ell_2} = a^{k_1+k_2}\,b^{\ell_1+\ell_2} = a^k b^\ell$$

where $k = k_1+k_2$, $\ell = \ell_1+\ell_2$. Here if $\ell > 0$, then $xy \in P$.

If $\ell = \ell_1+\ell_2 = 0$, then $\ell_1 = \ell_2 = 0$ and $k = k_1+k_2 > 0$. Hence xy P .

Clearly $e \notin P$.

Next we must show that P is a full right order, that is, $G/\{e\} = P \cup P^{-1}$. To see this, let $g \in G$, $g = a^n b^m$.

If $m > 0$, then $g \in P$.

If $m = 0$ and $n > 0$, then $g \in P$.

If $m = 0$ and $n = 0$, then $g = e$.

If $m = 0$ and $n < 0$, then $g^{-1} \in P$.

If $m < 0$, then $g^{-1} \in P$.

Further in this ordering $a << b$. (Notation: $a << b \iff a$ is infinitely smaller than b, that is, $a^n < b$ for every integer n.)

Now $b^2 \in Z(G)$, $b^2 > e$. Also, G is convexly generated by b^2, since $b^{m+1} > a^n b^m$ for every $n,m \in Z$. Thus by example 3 in 1.5 $G/\{b^2\} = K$, say is right cyclically ordered.

Now $K = \{a,b : b^{-1}ab = a^{-1}, b^2 = 1\}$ is the infinite dihedral group whose centre is trivial. K has torsion elements. Further, if T denotes the set of all torsion elements of k, then T is not even a subgroup of G. But the periodic elements of cyclically ordered groups always form a subgroup of the centre (see Fuchs [3], p. 63). Hence K cannot be cyclically ordered.

Thus as seen above the periodic elements in a right cyclically ordered group do not form a subgroup in general. However when they do form a subgroup it is central. This follows trivially from the following more general theorem:

Theorem 2.7: The periodic elements of a right cyclically ordered group $K = G/\{z\}$ form a subgroup if and only if the isolator of z is contained in the centre of G.

Proof: Let $I(z)$ denote the isolator of z and assume that $I(z) \subseteq Z(G)$. Let $\bar{x}$ be any periodic element of $K \cong G/\{z\}$. Thus $x \in G$ and for some integer n, $x^n \in \{z\}$. Hence $x \in I(z) \subseteq Z(G)$ and $\bar{x} \in Z(G)/\{z\} = Z(k)$.

This proves that the periodic elements of K lie in the centre of K and hence form a subgroup.

Conversely, let T denote the set of all periodic elements of K and assume that T forms a subgroup of K.

Consider $H = \{x \in G : x^n \in \{z\} \quad , \quad 0 \neq n \in Z\}$. Clearly

$$H/\{z\} \cong T \quad .$$

Thus H forms a subgroup of G. Since T is periodic right cyclically ordered it is abelian and locally cyclic.

Since $\{z\} \subseteq Z(H)$ and $H/\{z\}$ is locally cyclic, H is abelian.

Clearly H is normal in G. If H is not central, then there is $y \in G$ such that $x^y \neq x$, for some $x \in H$. Now since $x, x^y \in H$, the group $G_1 = \langle x, x^y \rangle$ is finitely generated abelian group. By the fundamental theorem of finitely generated abelian groups, G_1 is the direct sum of cyclic groups. Since G_1 is torsion-free, every factor is infinite cyclic. Hence, since G_1 is generated by two elements, $G_1 \cong C_\infty$ or $G_1 \cong C_\infty \times C_\infty$ (C_∞ denotes infinite cyclic group).

But G_1 cannot be cyclic. For, if G_1 is generated by a, then $x = a^n$, $x^y = a^m$ for some integers m and n. Since $x \in H$ there is some integer $k \neq 0$ such that $x^k \in \{z\}$. Thus

$$x^k = y^{-1}x^k y = (y^{-1}xy)^k = (x^y)^k \quad .$$

Hence $a^{nk} = x^k = (x^y)^k = a^{mk}$. Thus $a^{nk-mk} = e$ giving $n = m$ – a contradiction since $x \neq x^y$.

Thus x, x^y are free generators of abelian group G_1.

Thus $G_1 \cong C_\infty \times C_\infty$.

But since $x \in H$, there is $k \neq 0$ such that $x^k \in \{z\}$. Hence $x^k = (x^k)^y = (x^y)^k$ which is impossible. This contradicts that $G_1 \cong C_\infty \times C_\infty$.

Thus for every $y \in G$, and for every $x \in H$, $x^y = x$. Hence $H \subseteq Z(G)$. Since $I(z) = H$, the proof of the theorem is complete.

Corollary 2.8: Let K be a right cyclically ordered group, G be an ro-group such that $K = G/\{z\}$ (z in the centre of G). If G is an R-group then the periodic elements of K form a subgroup.

Proof: Immediate from Theorem 2.7.

(For R-groups, see Kurosh [4] §66, p. 242.)

SECTION III

The direct product of an ordered group and a cyclically ordered group is cyclically ordered. An analogous result for right cyclically ordered groups is:

Theorem 3.1: The direct product of a right ordered group and a right cyclically ordered group is right cyclically ordered.

Proof: Let H be a right ordered group. Then H is right cyclically ordered by the induced order from its right order, that is

$$[x,y,z] \iff \begin{array}{l} x < y < z \\ \text{or } \; y < z < x \\ \text{or } \; z < x < y \;. \end{array}$$

Let K be a right cyclically ordered group. Then $K \cong L/\{z\}$ where L is right ordered, $\{z\} \subseteq Z(L)$ and the convex subgroup generated by z is L.

Let $D = H \times L$. Right order D lexicographically so that any $\ell > e (\ell \in L)$ is infinitely greater than any $h \in H$. [Or: Define a positive cone P of D by $P = \{\langle h,\ell\rangle : \ell > e \text{ , or, } \ell = e \text{ and } h > e\}$] This ensures that the convex subgroup generated by $\langle e,z\rangle$ is D.

Let $G = D/\{\langle e,z\rangle\}$ ($\cong H \times K$). But G is right cyclically ordered and hence the theorem.

Now let Γ be an ordered group, and C the group of complex roots of unity under multiplication.

Let $[x,y,z]$ denote the induced cyclic order on Γ and (a,b,c) denote the cyclic order on C.

Then $\Gamma\times C$ can be cyclically ordered by:

$$[\langle x,a\rangle , \langle y,b\rangle , \langle z,c\rangle] = \begin{cases} (a,b,c) & \text{if} \quad a \neq b \neq c \neq a \\ x < y & \text{if} \quad a = b \neq c \\ y < z & \text{if} \quad b = c \neq a \\ z < x & \text{if} \quad c = a \neq b \\ [x,y,z] & \text{if} \quad a = b = c \quad . \end{cases}$$

(Swierczkowski [7] calls this a natural cyclic order.)

In fact, Swierczkowski [7] has proved that every cyclically ordered group is a subgroup of $\Gamma\times C$ for some ordered group Γ, the cyclic order on $\Gamma\times C$ being the natural cyclic order which induces on the given group its original cyclic order.

However, in general this can not happen in right cyclically ordered groups, since the torsion-part of a right cyclically ordered group is not as 'good' as that of a cyclically ordered group. Even when the torsion elements of a right cyclically ordered group form a subgroup, we do not have the above situation. This we show by the following example of a torsion-free right cyclically ordered group which cannot be right ordered.

Example 3.2: Let N be the free nilpotent group of class 2 with two generators, say $N = \{a,b : [a,b] = z \ , \ az = za \ , \ bz = zb\}$. The mapping: $a \to b$, $b \to (ab)^{-1}$ defines an automorphism of N of order 3.

$$\left(\begin{array}{l} \text{For,} \quad a \to b \to (ab)^{-1} = b^{-1}a^{-1} \to ab\cdot b^{-1} = a \\ \text{and} \quad b \to (ab)^{-1} = b^{-1}a^{-1} \to ab\cdot b^{-1} = a \to b \end{array}\right)$$

Hence we can define a group G :

$$G = \{c,N : c^{-1}ac = b \ , \ c^{-1}bc = (ab)^{-1}\} \quad .$$

This group G can be right ordered (For, N being a free nilpotent group is an 0-group. Also G/N is isomorphic to the infinite cyclic group and hence an 0-group (See Conrad [2], Theorem 3.7, p. 271).) The right ordering under reference can be described as follows:

Let P be a positive cone of N and P' that of G/N . We define the positive cone Q of G by

$$Q = \{e \neq g \in G : \text{ either } \ g \in N \cap P \ \text{ or } \ g \in G/N \ \text{ and } \ \overline{g} \in P'\} \quad .$$

Now $c^3 \in Z(G)$ and hence $c^9 z^{-1} = z^*$ (say) $\in Z(G)$. Then G is convexly generated by z^* , since $a << c$ and $b << c$ which holds as N is a convex subgroup of G with respect to the right order Q . Note that $z^* > e$. Hence $G/\{z^*\} = \underline{K}$ <u>say is right cyclically ordered</u>. Also, $\underline{K}$ <u>is torsion-free</u> (See Baumslag, Karrass and Solitar [1]).

Now N is naturally embedded in K, $[K:N] = 9$, $[K:K'] = 27$ and K is finitely generated. Hence K cannot be right ordered (See Rhemtulla [5], Theorem 1).

We have thus obtained a torsion-free right cyclically ordered group K which cannot be right ordered.

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